

How general is the global density slope–anisotropy inequality?

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ABSTRACT

Following the seminal result of An & Evans, known as the central density slope–anisotropy theorem, successive investigations unexpectedly revealed that the density slope–anisotropy inequality holds not only at the center, but at all radii in a very large class of spherical systems whenever the phase–space distribution function is positive. In this paper we derive a criterion that holds for all spherical systems in which the augmented density is a separable function of radius and potential: this new finding allows to unify all the previous results in a very elegant way, and opens the way for more general investigations. As a first application, we prove that the global density slope–anisotropy inequality is also satisfied by all the explored additional families of multi–component stellar systems. The present results, and the absence of known counter–examples, lead us to conjecture that the global density slope–anisotropy inequality could actually be a universal property of spherical systems with positive distribution function.

Key words: celestial mechanics – stellar dynamics – galaxies: kinematics and dynamics

1 INTRODUCTION

In the study of stellar systems based on the “ ρ -to- f ” approach (where ρ is the density distribution and f is the associated phase–space distribution function, hereafter DF), ρ is given, and specific assumptions on the internal dynamics of the model are made (e.g. see Bertin 2000; Binney & Tremaine 2008, hereafter BT08). In some special cases inversion formulae exist and the DF can be recovered in integral form or as series expansion (see, e.g., Fricke 1952; Lynden–Bell 1962; Osipkov 1979; Merritt 1985; Dejonghe 1986, 1987; Cuddeford 1991; Hunter & Qian 1993; Ciotti & Bertin 2005). Once the DF of the system is known, a non–negativity check should be performed, and in case of failure the model must be discarded as unphysical, even if it provides a satisfactory description of data. Indeed, a minimal but essential requirement to be met by the DF (of each component) of a stellar dynamical model is positivity over the accessible phase–space. This requirement, the so–called *phase–space consistency*, is much weaker than the model stability, but it is stronger than the fact that the Jeans equations have a physically acceptable solution. However, the difficulties inherent in the operation of recovering analytically the DF prevent in general a simple consistency analysis.

Fortunately, in special circumstances phase–space consistency can be investigated without an explicit recovery of the DF. For example, analytical necessary and sufficient conditions for consistency of spherically symmetric multi–component systems with Osipkov–Merritt anisotropy (Osipkov 1979, Merritt 1985, hereafter OM) were derived in Ciotti & Pellegrini (1992, hereafter CP92; see also Tremaine et al. 1994) and applied in several investigations (e.g., Ciotti 1996, 1999; Ciotti & Lanzoni 1997; Ciotti & Morganti 2009, hereafter CM09; Ciotti, Morganti & de Zeeuw 2009). Moreover, in Ciotti & Morganti (2010, hereafter CM10a) analytical necessary and sufficient consistency criteria for the family of spherically symmetric, multi–component, generalized Cuddeford systems (which contain as very special cases constant anisotropy and OM systems) have been derived.

Another necessary condition for consistency, the focus of this paper, is the “central cusp–anisotropy theorem” (An & Evans 2006, hereafter AE06; see also equation [28] in de Bruijne et al. 1996), an inequality relating the values of the central density slope γ and of the central anisotropy parameter β of consistent spherical systems, namely $\gamma \geq 2\beta$ (see Section 2). This condition was derived for constant anisotropy systems, and then generalized asymptotically to the central regions of spherical systems with arbitrary anisotropy distribution. In AE06 it was also shown that the density slope–anisotropy inequality actually holds rigorously at *every* radius in constant anisotropy systems, and

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not only at their center. We will refer to this case, i.e. when $\gamma(r) \geq 2\beta(r) \quad \forall r$, as the *Global Density–Slope Anisotropy Inequality* (hereafter GDSAI).

Surprisingly, in CM09 we showed that the CP92 necessary condition for model consistency is nothing else than the GDSAI in disguise. Thus, not only in constant anisotropy systems but also in each component of multi-component OM systems the GDSAI holds. Prompted by this curious result, in CM10a we introduced the larger family of multi-component generalized Cuddeford systems, we studied their phase-space consistency, and we finally proved that the GDSAI is again a necessary condition for phase-space consistency of each density component.

The results of CM09 and CM10a revealed an unexpected generality of the GDSAI and, in absence of known counter-examples (see also the discussion in CM10a), it is natural to ask whether the GDSAI is even more general, i.e. it is necessarily obeyed by all spherically symmetric, two-integrals systems with positive DF. If such conjecture proved true, it would be remarkable not only from a theoretical point of view, but also for applications. In fact, as it would hold separately for each density component of a stellar system, the value of the anisotropy parameter of the stellar component would be controlled at each radius by the local stellar density slope itself, independently of the dark matter halo. This constraint could be then used to reduce the impact of mass–anisotropy degeneracy in observational works¹. Motivated by the above arguments, we searched for a proof of the general validity of the GDSAI. While a proof is still missing, some relevant advance has been made: in particular, we obtained a new criterion that allows us not only to prove in a very elegant and unified way all our previous results, but also to demonstrate that new families of models do necessarily obey the GDSAI when consistent.

The paper is organized as follows. In Section 2 a general criterion linking phase-space consistency to the GDSAI for systems whose augmented density is a separable function of radius and potential is derived, which unifies all the results obtained so far on the GDSAI, and opens the way to the investigation of an even wider class of models. In Section 3 the new criterion is used to prove in a new way that the GDSAI is obeyed by generalized Cuddeford models, but also by some well-known stellar systems not belonging to the family of generalized Cuddeford models, thus further extending the validity of the GDSAI as a necessary condition for phase-space consistency. Finally, the main conclusions are summarized in Section 4.

2 A GENERAL CRITERION

In Ciotti & Morganti (2010, hereafter CM10b) we showed analytically, by direct computation, that two well-known anisotropic models not belonging to the generalized Cuddeford family, namely the Dejonghe (1987) anisotropic Plummer model, and the Baes & Dejonghe (2002) anisotropic Hernquist model, indeed obey the GDSAI whenever their

DF is positive. On one hand this result proves that the global inequality is not a specific property of generalized Cuddeford models, due perhaps to some special dependence of their DFs on energy and angular momentum. On the other hand, it reinforces the conjecture that the GDSAI could be a very general (if not a universal) property of consistent spherical models. In this Section we provide a new hint to the latter hypothesis, as we show, with the aid of a new criterion, how the GDSAI is rigorously valid in a very large class of consistent systems, containing not only the multi-component generalized Cuddeford systems and the two models discussed in CM10b, but whole new families of models.

2.1 General relations and the case of factorized systems

We consider stationary, non-rotating, spherically symmetric systems with a two-integrals phase-space distribution function $f = f(\mathcal{E}, J)$, where $\mathcal{E} = \Psi_T - v^2/2$ is the relative energy per unit mass, $\Psi_T = -\Phi_T$ is the relative total potential, and J is the angular momentum modulus per unit mass. In general Ψ_T may contain also the contribution of an “external” potential (for instance the one corresponding to a dark matter halo).

It is easy to show (e.g. see BT08, Ciotti 2000) that the density distribution ρ , the radial velocity dispersion σ_r , and the tangential velocity dispersion σ_t are related to the DF as

$$\rho = \frac{4\pi}{r^2} \int_0^{\Psi_T} d\mathcal{E} \int_0^{J_m} \frac{f(\mathcal{E}, J) J}{\Delta} dJ, \quad (1)$$

$$\rho \sigma_r^2 = \frac{4\pi}{r^2} \int_0^{\Psi_T} d\mathcal{E} \int_0^{J_m} f(\mathcal{E}, J) \Delta J dJ = p_r, \quad (2)$$

$$\rho \sigma_t^2 = \frac{4\pi}{r^4} \int_0^{\Psi_T} d\mathcal{E} \int_0^{J_m} \frac{f(\mathcal{E}, J) J^3}{\Delta} dJ = p_t, \quad (3)$$

where $\Delta = \sqrt{2(\Psi_T - \mathcal{E}) - J^2/r^2}$ and $J_m = r\sqrt{2(\Psi_T - \mathcal{E})}$. With p_r and p_t we indicate the radial and tangential pressure, respectively.

Given the identities above, and given the definition of the anisotropy parameter β , easy algebra proves the remarkable identities

$$\rho(r, \Psi_T) = \frac{\partial p_r}{\partial \Psi_T}, \quad (4)$$

$$\beta(r, \Psi_T) \equiv 1 - \frac{p_t}{2p_r} = -\frac{1}{2} \frac{\partial \ln p_r}{\partial \ln r} \quad (5)$$

(e.g., see Spies & Nelson 1974, Dejonghe 1986, Cavenago 1987, Dejonghe & Merritt 1992, Bertin et al. 1994, Bertin 2000, Baes & van Hese 2007). Note that these relations hold independently of the specific radial dependence of Ψ_T , and that the radial trends of ρ and β are known only after the total potential Ψ_T is given. The existence of the general relations (4) and (5) is of the greatest importance for the present study, because it shows clearly that ρ (and so γ) and β are somewhat linked by the function p_r . This link could open the way to a general proof of the GDSAI.

At the present stage, we do not attempt a general proof, but we focus our attention on a special case, that of spherical systems with a factorized augmented density. In practice, we consider special spherical systems in which the radial

¹ Clearly, as by definition $\beta \leq 1$, such limitation would be useful only in the regions where the density slope is ≤ 2 , while in the galaxy outskirts the inequality is not helpful, as $\gamma > 2$ for mass convergence.

pressure p_r is a factorized function of the radius and of the total potential. Therefore, from equation (4), it follows that also the density distribution can be written as

$$\rho(r, \Psi_T) \equiv A(r)B(\Psi_T); \quad (6)$$

of course, while the function B in the expression above is the derivative of the potential dependent factor in the factorized expression of p_r , the radial function A is the same. We note that the multi-component generalized Cuddeford models (and therefore also the constant anisotropy, OM, and Cuddeford models, see CM10a), as well as the two models discussed in CM10b, belong to such family of systems. From equation (5) it follows that, independently of the specific radial dependence of Ψ_T ,

$$2\beta(r) = -\frac{d \ln A}{d \ln r}. \quad (7)$$

Now, for assigned $\Psi_T(r)$, equation (4) shows that the logarithmic slope of the density profile (6) can be written as

$$\gamma(r) \equiv -\frac{d \ln \rho}{d \ln r} = -\frac{d \ln B}{d \ln \Psi_T} \frac{d \ln \Psi_T}{d \ln r} + 2\beta(r); \quad (8)$$

since from Newton theorem $\Psi_T(r)$ is a monotonically decreasing function of radius, identity (8) proves the following

Criterion 1: in all spherical systems whose density distribution is a separable function of radius and total potential, $\rho = A(r)B(\Psi_T)$, the global inequality $\gamma(r) \geq 2\beta(r) \quad \forall r$ holds $\Leftrightarrow B(\Psi_T)$ is a monotonically increasing function of Ψ_T .

Therefore, if one is able to show that in all factorized consistent systems the B function is necessarily monotonic, then the GDSAI will hold in all these systems. We were not able to prove or disprove this possibility in general, however in Section 3 we present interesting results along this line.

Before moving to discuss the new results, we note the following curious fact: the GDSAI can also be expressed as a condition on the radial velocity dispersion of the model. In fact, for a two-integrals spherical system, the relevant Jeans equation can be written as

$$\frac{d\rho\sigma_r^2}{dr} + \frac{2\beta\rho\sigma_r^2}{r} = \rho \frac{d\Psi_T}{dr} \quad (9)$$

(e.g., BT08). Introducing the logarithmic density slope as in equation (8), and rearranging the terms, one finds

$$\gamma(r) - 2\beta(r) = r \left(\frac{d\sigma_r^2}{dr} - \frac{d\Psi_T}{dr} \right) \geq 0 \quad (10)$$

as an equivalent, alternative formulation of the GDSAI. Unfortunately, despite the deceptively simple form, for a given family of consistent models proving the necessity of the GDSAI from equation (10) is not easier than working directly on phase-space.

3 RESULTS

In this Section we apply Criterion 1 to two families of models, and we show that also these systems not belonging to the generalized Cuddeford family obey the GDSAI in case of phase-space consistency. However, in order to illustrate the power of the new result, with the aid of Criterion 1 we first re-derive almost immediately the results obtained in

CM10a with some lengthy algebra for the generalized Cuddeford models.

3.1 A new proof of the GDSAI for generalized Cuddeford systems

As described in CM10a, the DF of each component of generalized Cuddeford systems is described by the sum of an arbitrary number of Cuddeford (1991) DFs with arbitrary positive weights w_i ,

$$f = J^{2\alpha} \sum_i w_i h(Q_i), \quad Q_i = \mathcal{E} - \frac{J^2}{2r_{ai}^2}, \quad (11)$$

and possibly different anisotropy radii r_{ai} , but same h function and angular momentum exponent α . Here, α is a real number > -1 , and $h(Q_i) = 0$ for $Q_i \leq 0$. Note that multi-component OM models (CP92, CM09), and constant anisotropy models are special cases of equation (11), obtained for $i = 1$ and $\alpha = 0$, and $i = 1$ and $r_a = \infty$, respectively.

As shown in CM10a, the spatial density associated with the DF (11) is a factorized function as in equation (6), where

$$A(r) = (2\pi)^{3/2} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)} \sum_i \frac{w_i r^{2\alpha}}{(1+r^2/r_{ai}^2)^{\alpha+1}}, \quad (12)$$

$$B(\Psi_T) = \int_0^{\Psi_T} (\Psi_T - Q)^{\alpha+1/2} h(Q) dQ, \quad (13)$$

and $\Gamma(x)$ is the complete gamma function. Therefore, the anisotropy parameter β , derived in CM10a by direct computation, can now be obtained from equations (12) and (7), and Criterion 1 can be applied to the family of generalized Cuddeford models. In CM10a it was shown with some algebra (see equations [19] and [25] therein) that for $\alpha \geq -1/2$ the first of the necessary conditions for phase-space consistency can be rewritten as the GDSAI. A more elegant proof can now be obtained by using Criterion 1. In fact, direct differentiation of equation (23) shows that B is a monotonically increasing function of Ψ_T whenever $h > 0$ and $\alpha \geq -1/2$. An interesting situation arises for $\alpha = -1/2$. In this case equation (28) in CM10a shows that the GDSAI is both necessary and sufficient condition for phase-space consistency: in the present context $dB/d\Psi_T = h(\Psi_T)$, and so the monotonicity of B is ensured if and only if h is positive, confirming the CM10a result. We are left with the case $-1 < \alpha < -1/2$. As shown in CM10a, in this case only a sufficient condition, coincident with the GDSAI, is available. What are the information that can be derived from Criterion 1 in this case? Indeed, for $-1 < \alpha < -1/2$, it is not possible to compute the derivative of equation (23) directly, and so to test the monotonicity of B . However, this problem can be circumvented by first integrating by parts, and then performing differentiation, obtaining

$$\frac{dB}{d\Psi_T} = \Psi_T^{\alpha+1/2} h(0) + \int_0^{\Psi_T} (\Psi_T - Q)^{\alpha+1/2} h'(Q) dQ. \quad (14)$$

From the expression above, we conclude that the GDSAI is again necessary for phase-space consistency in all generalized multi-component Cuddeford systems with $-1 < \alpha < -1/2$ and $h'(Q) > 0$.

Summarizing, for $\alpha > -1/2$ the GDSAI is necessary for consistency, for $\alpha = -1/2$ it is equivalent (i.e., necessary and sufficient), and for $-1 < \alpha < -1/2$ it is just sufficient, but also necessary if $h'(Q) > 0$. Of course, whenever the last condition is not satisfied, the possibility to build a counter-example to the conjecture that the GDSAI is a universal *necessary* condition for consistency remains open. If one could prove that for $-1 < \alpha < -1/2$ the inequality is necessary whenever $h > 0$, then one would conclude that in multi-component generalized Cuddeford models the GDSAI is equivalent to consistency for $-1 < \alpha \leq -1/2$.

3.2 The Baes & van Hese (2007) anisotropic models

By using Criterion 1 we now show that the GDSAI is obeyed by another (quite large) family of model.

Baes & van Hese (2007, Section 4.2) considered the family of augmented density profiles

$$\rho = \rho_0 \left(\frac{r}{r_a} \right)^{-2\beta_0} \left(1 + \frac{r^{2\delta}}{r_a^{2\delta}} \right)^{\frac{\beta_0 - \beta_\infty}{\delta}} \left(\frac{\Psi}{\Psi_0} \right)^p \left(1 - \frac{\Psi^s}{\Psi_0^s} \right)^q, \quad (15)$$

where $\delta > 0$, $q \leq 0$, $s > 0$, $0 \leq \Psi \leq \Psi_0$, and Ψ_0 is the value of the central relative potential. From equation (7) the anisotropy profile is

$$\beta(r) = \frac{\beta_0 + \beta_\infty (r/r_a)^{2\delta}}{1 + (r/r_a)^{2\delta}}, \quad (16)$$

so that β_0 and β_∞ are the values of the orbital anisotropy parameter at small and large radii, respectively; since the anisotropy parameter β cannot exceed the value of 1 in case of positive DF, both β_0 and β_∞ are ≤ 1 .

Note that the requirement of finite total mass for the density distribution (15), i.e. $\Psi(r) \sim 1/r$ for $r \rightarrow \infty$, translates into the condition $p + 2\beta_\infty > 3$, and from the limitation on β_∞ it follows that $p \geq 1$. Now, it is easy to show that $dB(\Psi)/d\Psi > 0$ for $0 \leq \Psi \leq \Psi_0$, when $s > 0$, $q \leq 0$, and $p \geq 1$, and thus Criterion 1 ensures that the GDSAI is necessarily obeyed also by these profiles. In the particular case $q = 0$, it is possible to prove again the GDSAI by using equation (10) and considering the expression of the radial velocity dispersion given in equation (37) of Baes & van Hese (2007). Note also that the analytical DF associated to these systems is

$$f = \sum_k \mathcal{E}^{p+ks-3/2} g_k \left(\frac{J^2}{2\mathcal{E}} \right), \quad (17)$$

where the g_k are hypergeometric functions (see equations [41] and [42] in Baes & van Hese 2007), and so in this case the DF is not of the Cuddeford generalized family.

We finally note that the two models discussed analytically in CM10b, i.e. the Dejonghe (1987) anisotropic Plummer model, and the Baes & Dejonghe (2002) anisotropic Hernquist model, are both special cases of the family (15), and therefore the fact that they also obey the GDSAI is just a special case of the result of this Section.

3.3 The Cuddeford & Louis (1995) anisotropic polytropes

We finally consider the family of models introduced by Cuddeford & Louis (1995). At variance with the previous models, these systems are not introduced by using the augmented density technique, but from their DF, in a way similar to the models in Section 3.1, so that Criterion 1 cannot be applied directly. Their DF is

$$f(\mathcal{E}, J) = \mathcal{E}^{q-2} h(k), \quad k \equiv \frac{J^2}{2r_a^2 \mathcal{E}}, \quad (18)$$

where $\mathcal{E} \geq 0$, and r_a is the anisotropy radius. These models are known as *anisotropic polytropes*, and the special case $h(k) = (1+k)^\alpha$ was studied by Louis (1993). The formulae for ρ and p_r were already obtained also for the more general case of $f = g(\mathcal{E})h(k)$, and here we just report the result for the case (18):

$$\rho = 2^{3/2} \pi B(q, 1/2) \Psi_T^{q-1/2} \eta^{q-1} \int_0^\infty \frac{h(k) dk}{(k+\eta)^q}, \quad (19)$$

$$p_r = 2^{5/2} \pi B(q, 3/2) \Psi_T^{q+1/2} \eta^{q-1} \int_0^\infty \frac{h(k) dk}{(k+\eta)^q}, \quad (20)$$

where B is the complete Beta function and $\eta = r^2/r_a^2$. Incidentally, the validity of equations (19) and (20) can be easily checked by using equation (4). Note that the convergence of the energy integral in equation (19) requires $q > 0$ near $\mathcal{E} = 0$. As p_r and ρ are in factorized form (see also Dejonghe 1986, Sect. 1.7.3), we can apply Criterion 1: since $\rho \propto \Psi_T^{q-1/2}$, the GDSAI is satisfied whenever $q \geq 1/2$. This result can be obtained immediately also from equation (10), as the radial velocity dispersion has the remarkably simple expression

$$\sigma_r^2 = \frac{p_r}{\rho} = \frac{2\Psi_T}{2q+1}. \quad (21)$$

The situation is less straightforward in the interval $0 < q < 1/2$. In fact, if a consistent model exists in this interval, then it will represent a case of violation of the GDSAI. Therefore, it is natural to ask whether it is possible to construct a consistent dynamical model with $q > 0$, but violating the GDSAI ($q < 1/2$). We are not able to answer this question in general, however we note that in the self-consistent case of finite total mass in which $\Psi_T \sim 1/r$ for $r \rightarrow \infty$, volume integration of equation (19) and successive inversion of order of integration shows that $q > 3/2$ is necessary in order to have a finite mass for the component under scrutiny (with possible further restrictions dependent on the asymptotic nature of the h function). In other words, there are not consistent anisotropic polytropes with finite mass and $q < 3/2$. Therefore the GDSAI is a necessary condition for anisotropic polytropes in generic external potential when $q \geq 1/2$, and for all finite mass self-consistent models. It remains open the possibility of existence of consistent anisotropic polytropes (of infinite mass) with $0 < q < 1/2$, which would violate the GDSAI.

Actually the discussion above can be generalized, similarly to what we did in CM10a when we constructed the family of generalized Cuddeford anisotropic systems. In fact, we now consider the family of generalized anisotropic polytropes with DF

$$f(\mathcal{E}, J) = \mathcal{E}^{q-2} \sum_i w_i h(k_i), \quad k_i \equiv \frac{J^2}{2r_{\text{ai}}^2 \mathcal{E}}, \quad (22)$$

with different anisotropy radii r_{ai} and positive weights w_i . With the definition $\eta_i = r^2/r_{\text{ai}}^2$, equations (19) and (20) become

$$\rho = 2^{3/2} \pi B(q, 1/2) \Psi^{q-1/2} \sum_i w_i \eta_i^{q-1} \int_0^\infty \frac{h(k) dk}{(k + \eta_i)^q}, \quad (23)$$

$$p_r = 2^{5/2} \pi B(q, 3/2) \Psi^{q+1/2} \sum_i w_i \eta_i^{q-1} \int_0^\infty \frac{h(k) dk}{(k + \eta_i)^q}, \quad (24)$$

and it is immediate to see that all the previous conclusions hold also for these general models, even though their DF is not of the family (18).

4 DISCUSSION AND CONCLUSIONS

After the discovery of the “central cusp–anisotropy theorem” (AE06), an inequality relating the central value of the density slope and the anisotropy parameter in consistent stellar systems, successive investigations (CM09, CM10a) unexpectedly revealed that the density slope–anisotropy inequality holds not only at the center, but at all radii in a very large class of spherical systems (the generalized multi-component Cuddeford systems), whenever the phase-space distribution function is positive. We call this latter inequality the *Global Density Slope Anisotropy Inequality* (GDSAI).

In absence of known counter-examples, i.e. two-integrals stellar systems with positive DF but violating the GDSAI, in this paper we focused on the possibility that the GDSAI is actually universal, i.e. it is necessarily obeyed by all spherically symmetric, two-integrals systems with positive DF. If such conjecture proved true, it would be remarkable not only from a theoretical point of view, but also for applications. In fact, it could be used to reduce the impact of mass–anisotropy degeneracy in observational works, as orbital anisotropy would be in some sense controlled by the local density slope of the stellar distribution in galaxies (in the inner regions where $\gamma \leq 2$). While a proof of this conjecture is still missing, some relevant advance has been made: in particular, we obtained a new criterion that allows us not only to prove in a simple, very elegant, and unified way all the previously known results, but also to investigate new families of multi-component models. The main results of this paper can be summarized as follows:

- (i) By using two previously known and fully general identities relating the density profile, the anisotropy profile, and the radial pressure in two-integrals systems, specialized to the case of factorized systems, we found a very simple condition equivalent to the GDSAI, namely that the potential dependent function in the augmented density is monotonically increasing.
- (ii) As a first application of the new condition, we showed that all the previous cases, each of them proved with “ad-hoc” analysis, are in fact all simple cases of the new relation, and the proof is almost immediate.
- (iii) The new criterion is then applied to extend the validity of the GDSAI to other models, namely the Baes & van Hese (2007) anisotropic models, and the Cuddeford & Louis (1995) anisotropic polytropes. This latter family is extended

to generalized multi-component anisotropic polytropes, and it is shown that also in this case the GDSAI holds.

(iv) As we are not able to show that the monotonicity of the potential dependent function in the factorized augmented density is necessarily monotonic whenever the DF is positive, our investigation leaves open the possibility that some spherical systems with positive phase-space distribution function may violate the GDSAI. Such examples, if they exist, may be found (in the class of models studied so far) only in the family of Cuddeford models with angular momentum exponent in the range $-1 < \alpha < -1/2$, or in the family of Cuddeford & Louis (1995) anisotropic polytropes with infinite mass and $0 < q < 1/2$.

Finally, we conclude by noticing that the proof of the general validity of the GDSAI could be obtained by using equations (4) and (5) in all their generality. At the present stage we do not have such a proof; however, neither counter-examples are known to us. Moreover, we note that supporting arguments are provided by numerical simulations of N-body systems, whose end-products show correlations between β and γ (e.g., see Hansen & Moore 2006, Mamon et al. 2006). In addition, Michele Trenti kindly provided us with a large set of numerically computed f_ν models (Bertin & Trenti 2003) and all of them, without exception, satisfy the GDSAI. We also verified numerically that the GDSAI is satisfied by a large set of radially anisotropic Hernquist and Jaffe models with quasi-separable DF constructed by Gerhard (1991) by using the so-called h_α circularity function.

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